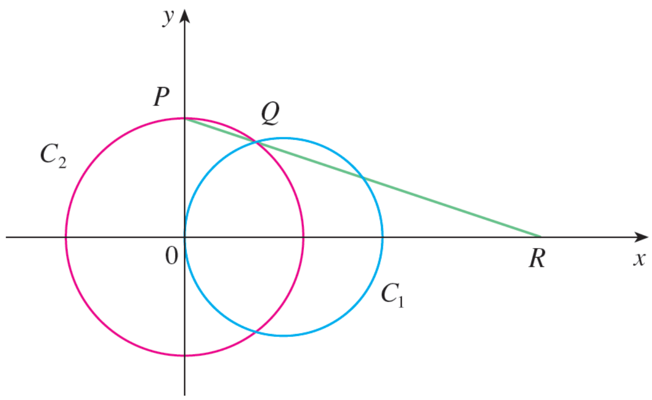
Exercise 66

The figure shows a fixed circle C_1 with equation $(x-1)^2 + y^2 = 1$ and a shrinking circle C_2 with radius r and center the origin. P is the point $(0, r)$, Q is the upper point of intersection of the two circles, and R is the point of intersection of the line PQ and the x-axis. What happens to R as C_2 shrinks, that is, as $r \to 0^+$?



Solution

The aim is to find the equation for the line PQ because once it's known, the x-intercept will be R. One point on this line that's known is P: $(0, r)$. The other point Q is unknown, but it can be found by solving the two equations for the circles simultaneously since Q is a point of intersection.

$$\left.\begin{aligned} x^2 + y^2 &= r^2 \\ (x-1)^2 + y^2 &= 1 \end{aligned}\right\}$$

Subtract the two equations to eliminate y^2. Then solve for x.

$$x^2 - (x-1)^2 = r^2 - 1$$

$$x^2 - (x^2 - 2x + 1) = r^2 - 1$$

$$2x - 1 = r^2 - 1$$

$$2x = r^2$$

$$x = \frac{r^2}{2}$$

Substitute this result back into either of the two equations to get the corresponding y-value.

$$x^2 + y^2 = r^2$$

$$\left(\frac{r^2}{2}\right)^2 + y^2 = r^2$$

Solve for y.

$$\frac{r^4}{4} + y^2 = r^2$$

$$y^2 = r^2 - \frac{r^4}{4}$$

$$y^2 = \frac{r^2}{4}(4 - r^2)$$

$$y = \pm\frac{r}{2}\sqrt{4 - r^2}$$

The positive value of y is taken since Q is the intersection above the x-axis. As a result, the point for Q is

$$\left(\frac{r^2}{2}, \frac{r}{2}\sqrt{4 - r^2}\right).$$

Determine the slope of line PQ.

$$m = \frac{\frac{r}{2}\sqrt{4 - r^2} - r}{\frac{r^2}{2} - 0} = \frac{\sqrt{4 - r^2} - 2}{r}$$

Use the point-slope formula with point P to get the equation of the line.

$$y - r = \frac{\sqrt{4 - r^2} - 2}{r}(x - 0)$$

$$y - r = \left(\frac{\sqrt{4 - r^2} - 2}{r}\right)x$$

The x-intercept of this line is $(R, 0)$.

$$0 - r = \left(\frac{\sqrt{4 - r^2} - 2}{r}\right)R$$

Solve for R.

$$R = \frac{r^2}{2 - \sqrt{4 - r^2}}$$

Finally, take the limit of R as $r \to 0^+$.

$$\begin{aligned}
\lim_{r\to 0^+} R &= \lim_{r\to 0^+} \frac{r^2}{2-\sqrt{4-r^2}} \\
&= \lim_{r\to 0^+} \frac{r^2}{2-\sqrt{4-r^2}} \times \frac{2+\sqrt{4-r^2}}{2+\sqrt{4-r^2}} \\
&= \lim_{r\to 0^+} \frac{r^2\left(2+\sqrt{4-r^2}\right)}{\left(2-\sqrt{4-r^2}\right)\left(2+\sqrt{4-r^2}\right)} \\
&= \lim_{r\to 0^+} \frac{r^2\left(2+\sqrt{4-r^2}\right)}{4-(4-r^2)} \\
&= \lim_{r\to 0^+} \frac{r^2\left(2+\sqrt{4-r^2}\right)}{r^2} \\
&= \lim_{r\to 0^+} \left(2+\sqrt{4-r^2}\right) \\
&= 2+\sqrt{4-0^2} \\
&= 4
\end{aligned}$$